YNCenter Video Conference

A new k-e model of turbulence and diffusion processes in shallow lakes

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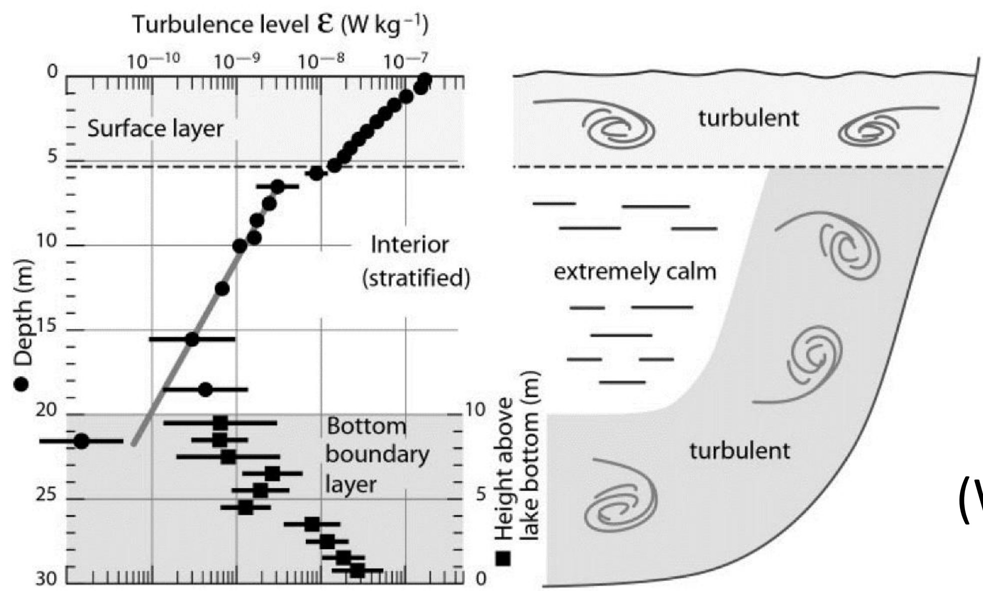
Outline

- Background
- Model structure
- Experimental Data
- Preliminary Result
- Summary

Background

Water movements ***forced by the wind*** produce turbulent mixing which combines with ***surface heating or cooling*** to determine the vertical structure of the lake and mediate the vertical fluxes of scalars which, in turn, have major impacts on lake ecology. (Woolway 2017)

Depending on the properties of the air, wind, water, and waves, the air-water interface creates a ***bottleneck*** for the exchange of the physical quantities such as heat, kinetic energy, momentum, and matter (gases, vapor, aerosols, etc.).



(Wüest and Lorke 2003)

Background

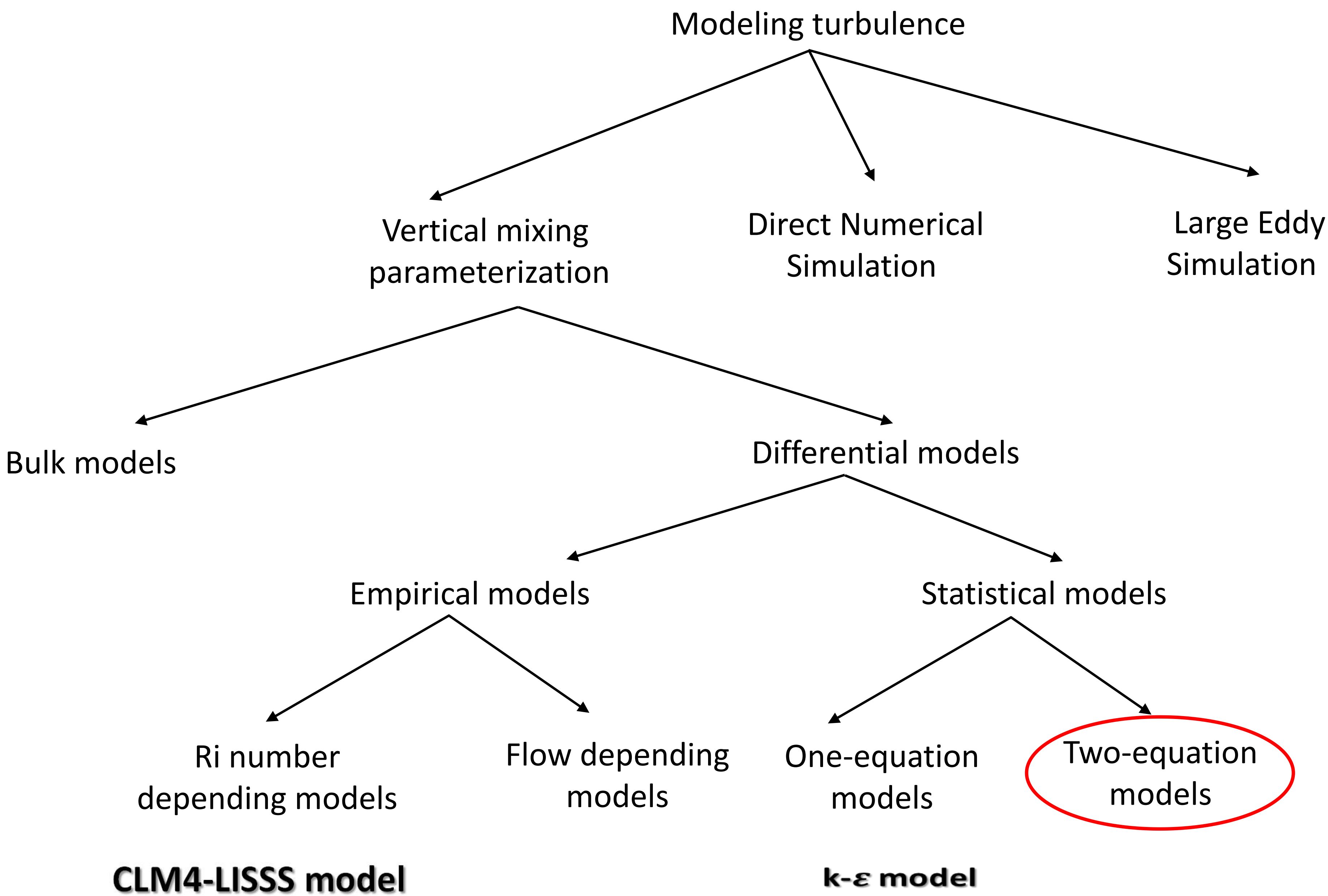
Complicate turbulence movement is difficult to be measured, developing a *mathematical description* of turbulent stresses sought to mimic the molecular gradient-diffusion process.

Transport equations for turbulent kinetic energy and turbulent dissipation rate can be derived by *Reynolds averaging*. For the TKE equation, closure assumptions have to be made only for *the turbulent transport terms*.

$$\partial_t k + \partial_z F(k) = K_M S^2 - K_H N^2 - \varepsilon$$

$$\partial_t \varepsilon + \partial_z F(\varepsilon) + = c_{\varepsilon 1} \frac{\varepsilon}{k} (K_M S^2 - c_{\varepsilon 3} K_H N^2) - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

(Burchard 2008)


Modeling turbulence
Vertical mixing parameterization
Direct Numerical Simulation
Large Eddy Simulation
Bulk models
Differential models
Empirical models
Statistical models
Ri number depending models
Flow depending models
One-equation models
Two-equation models
CLM4-LISSS model
k-ε model

Model Structure

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial u}{\partial z}] + \mathrm{A}c_b u^2 \qquad m/s$$

$$\frac{\partial \mathrm{T}}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial \mathrm{T}}{\partial z}] + \frac{1}{c_{liq}}\frac{d\phi}{dz} \qquad ℃$$

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial E}{\partial z}] + K_e(\frac{\partial u}{\partial z})^2 + \theta(T)gK_e\frac{\partial T}{\partial z} - \varepsilon \qquad m^2/s^2$$

Shearing production | Buoyancy term | Dissipation term

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial \varepsilon}{\partial z}] + [c_{1\varepsilon}K_e(\frac{\partial u}{\partial z})^2 + c_{3\varepsilon}\theta(T)gK_e\frac{\partial T}{\partial z} - c_{2\varepsilon}\varepsilon]\frac{\varepsilon}{k} \qquad m^2/s^3$$

$$K_e = c_\mu\frac{E^2}{\varepsilon} \qquad m^2/s$$

Step1 : Crank-Nicolson method $$\frac{\partial X}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial X}{\partial z}\right)+\phi$$

Step2 : Tridiagonal matrix algorithm $$A_iX_{i+1}^n+\mathrm{B}_iX_i^n+C_iX_{i-1}^n=D_i$$

$$\begin{pmatrix} B_1 & C_1 & 0 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & 0 & 0 & 0 \\ 0 & A_3 & B_3 & C_3 & 0 & 0 \\ 0 & 0 & A_4 & B_4 & C_4 & 0 \\ 0 & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \mathrm{A}_i & B_i \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ \vdots \\ T_i \end{pmatrix}=\begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \vdots \\ D_i \end{pmatrix}$$

$$\gamma_i=\begin{cases} \dfrac{C_i}{B_i} & ; \quad i=1 \\ \dfrac{C_i}{B_i-A_i\gamma_{i-1}} & ; i=2,3,\ldots,n-1 \end{cases}$$

$$\rho_i=\begin{cases} \dfrac{D_i}{B_i} & ; \quad i=1 \\ \dfrac{D_i-A_i\rho_i}{B_i-A_i\gamma_{i-1}} & ; i=2,3,\ldots,n-1 \end{cases}$$

Heat diffusivity equation

Differential equation version: $\frac{\partial \mathrm{T}}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial \mathrm{T}}{\partial z}] + \frac{1}{c_{liq}}\frac{d\phi}{dz}$

Tri-diagonal version: $a_i = -\frac{\Delta t}{\Delta z_i}(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}})^{-1}$

$$b_i = 1 + \frac{\Delta t}{\Delta z_i}(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}})^{-1} + \frac{\Delta t}{\Delta z_i}(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}})^{-1}$$

$$c_i = -\frac{\Delta t}{\Delta z_i}(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}})^{-1}$$

$$r_i = T_i^n + \frac{\Delta t}{\Delta z_i}(T_{i-1}^n - T_i^n)(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}})^{-1}$$

$$-\frac{\Delta t}{\Delta z_i}(T_i^n - T_{i+1}^n)(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}})^{-1} + \frac{\Delta t}{\Delta z_i}\frac{\phi_{i-\frac{1}{2}} - \phi_{i+\frac{1}{2}}}{c_{liq}}$$

Boundary condition: upper boundary: surface temperature

lower boundary: sediment temperature

Initial condition: observation water temperature

Momentum equation

Differential equation version: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left[(K_m + K_e)\frac{\partial u}{\partial z}\right] + \mathrm{A}c_b u^2$

Tridiagonal version: $a_i = -\frac{\Delta t}{\Delta z_i}\left(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}}\right)^{-1}$

$$b_i = 1 + \frac{\Delta t}{\Delta z_i}\left(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}}\right)^{-1} + \frac{\Delta t}{\Delta z_i}\left(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}}\right)^{-1}$$

$$c_i = -\frac{\Delta t}{\Delta z_i}\left(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}}\right)^{-1}$$

$$r_i = u_i^n + \frac{\Delta t}{\Delta z_i}(u_{i-1}^n - u_i^n)\left(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}}\right)^{-1}$$

$$-\frac{\Delta t}{\Delta z_i}(u_i^n - u_{i+1}^n)\left(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}}\right)^{-1} + \frac{\Delta t}{\Delta z_i}c_b(u_i^2 - u_{i-1}^2)$$

Boundary condition: upper boundary: surface friction velocity

lower boundary: close to zero

Initial condition: adjusted

Turbulent diffusivity equation

Differential equation version: $\frac{\partial E}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial E}{\partial z}] + K_e(\frac{\partial u}{\partial z})^2 + \theta(T)gK_e\frac{\partial T}{\partial z} - \varepsilon$

Tri-diagonal version:

$$a_i = -\frac{\Delta t}{\Delta z_i}(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}})^{-1}$$

$$b_i = 1 + \frac{\Delta t}{\Delta z_i}(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}})^{-1} + \frac{\Delta t}{\Delta z_i}(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}})^{-1}$$

$$c_i = -\frac{\Delta t}{\Delta z_i}(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}})^{-1}$$

$$\boldsymbol{K_e = c_\mu \frac{E^2}{\varepsilon}}$$

$$r_i = \mathrm{E}_i^n + \frac{\Delta t}{\Delta z_i}(\mathrm{E}_{i-1}^n - \mathrm{E}_i^n)(\frac{\Delta z_{i-1}}{K_m+K_{e,i-1}} + \frac{\Delta z_i}{K_m+K_{e,i}})^{-1}$$

$$-\frac{\Delta t}{\Delta z_i}(\mathrm{E}_i^n - \mathrm{E}_{i+1}^n)(\frac{\Delta z_i}{K_m+K_{e,i}} + \frac{\Delta z_{i+1}}{K_m+K_{e,i+1}})^{-1} + \frac{\Delta t}{\Delta z_i}(\phi_{i-\frac{1}{2}} - \phi_{i+\frac{1}{2}})$$

ϕ?

Boundary condition: upper boundary: u_*^2

lower boundary: close to zero

Initial condition: adjusted

Dissipation equation

Differential equation: $\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z}[(K_m + K_e)\frac{\partial \varepsilon}{\partial z}] + [c_{1\varepsilon} K_e (\frac{\partial u}{\partial z})^2 + c_{3\varepsilon}\theta(T) g K_e \frac{\partial T}{\partial z} - c_{2\varepsilon}\varepsilon]\frac{\varepsilon}{k}$

Tri-diagonal version:

$$a_i = -\frac{\Delta t}{\Delta z_i}(\frac{\Delta z_{i-1}}{K_m + K_{e,i-1}} + \frac{\Delta z_i}{K_m + K_{e,i}})^{-1}$$

$$b_i = 1 + \frac{\Delta t}{\Delta z_i}(\frac{\Delta z_{i-1}}{K_m + K_{e,i-1}} + \frac{\Delta z_i}{K_m + K_{e,i}})^{-1} + \frac{\Delta t}{\Delta z_i}(\frac{\Delta z_i}{K_m + K_{e,i}} + \frac{\Delta z_{i+1}}{K_m + K_{e,i+1}})^{-1}$$

$$c_i = -\frac{\Delta t}{\Delta z_i}(\frac{\Delta z_i}{K_m + K_{e,i}} + \frac{\Delta z_{i+1}}{K_m + K_{e,i+1}})^{-1}$$

$$\boldsymbol{K_e = c_\mu \frac{E^2}{\varepsilon}}$$

$$r_i = \varepsilon_i^n + \frac{\Delta t}{\Delta z_i}(\varepsilon_{i-1}^n - \varepsilon_i^n)(\frac{\Delta z_{i-1}}{K_m + K_{e,i-1}} + \frac{\Delta z_i}{K_m + K_{e,i}})^{-1}$$

$$-\frac{\Delta t}{\Delta z_i}(\varepsilon_i^n - \varepsilon_{i+1}^n)(\frac{\Delta z_i}{K_m + K_{e,i}} + \frac{\Delta z_{i+1}}{K_m + K_{e,i+1}})^{-1} + \frac{\Delta t}{\Delta z_i}(\phi_{i-\frac{1}{2}} - \phi_{i+\frac{1}{2}})$$

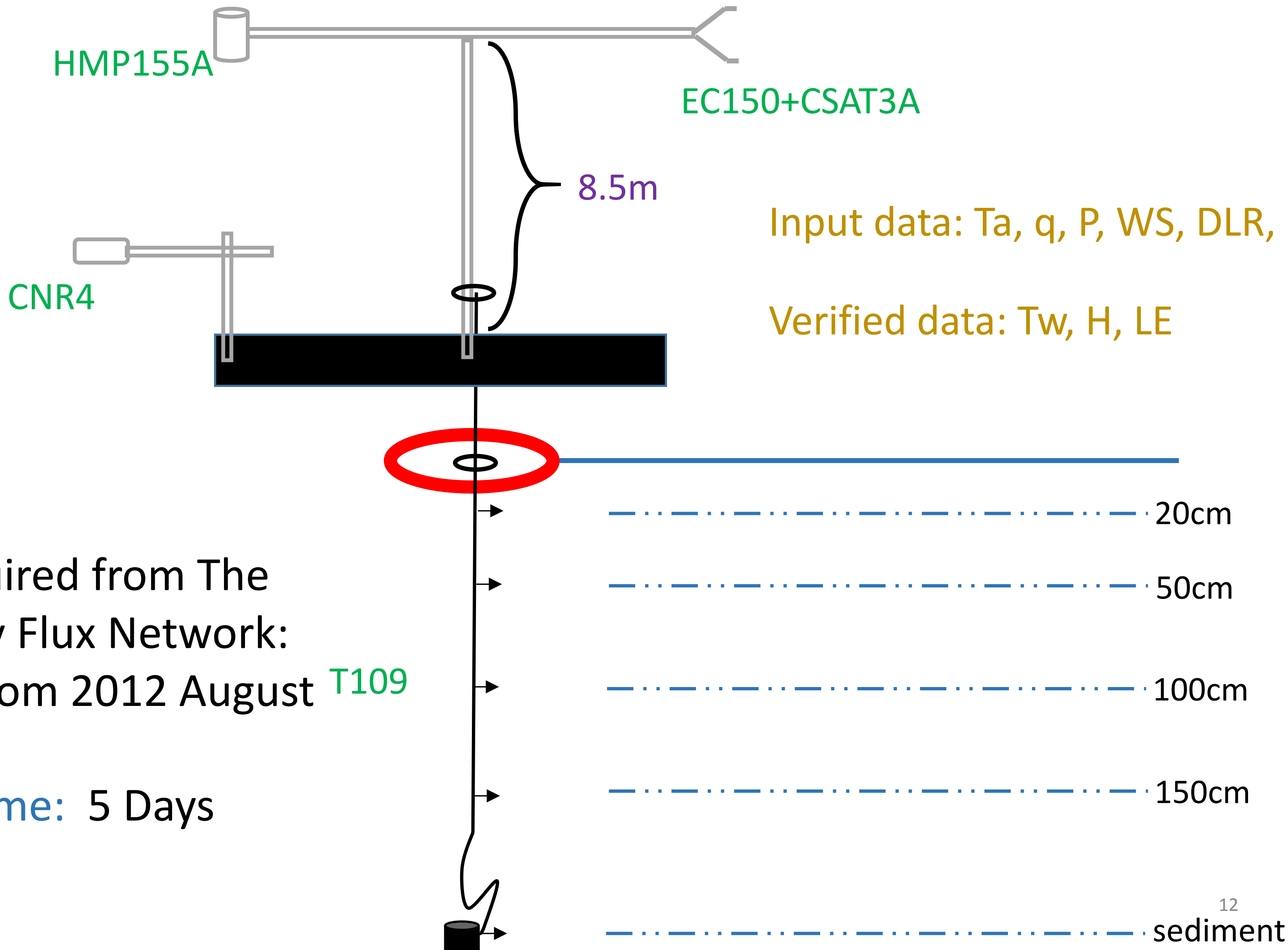
ϕ?

Boundary condition: upper boundary: $c_\mu \frac{E^2}{K_e}$

lower boundary: close to zero

Initial condition: adjusted

Experimental Data



Input data: Ta, q, P, WS, DLR, Rn

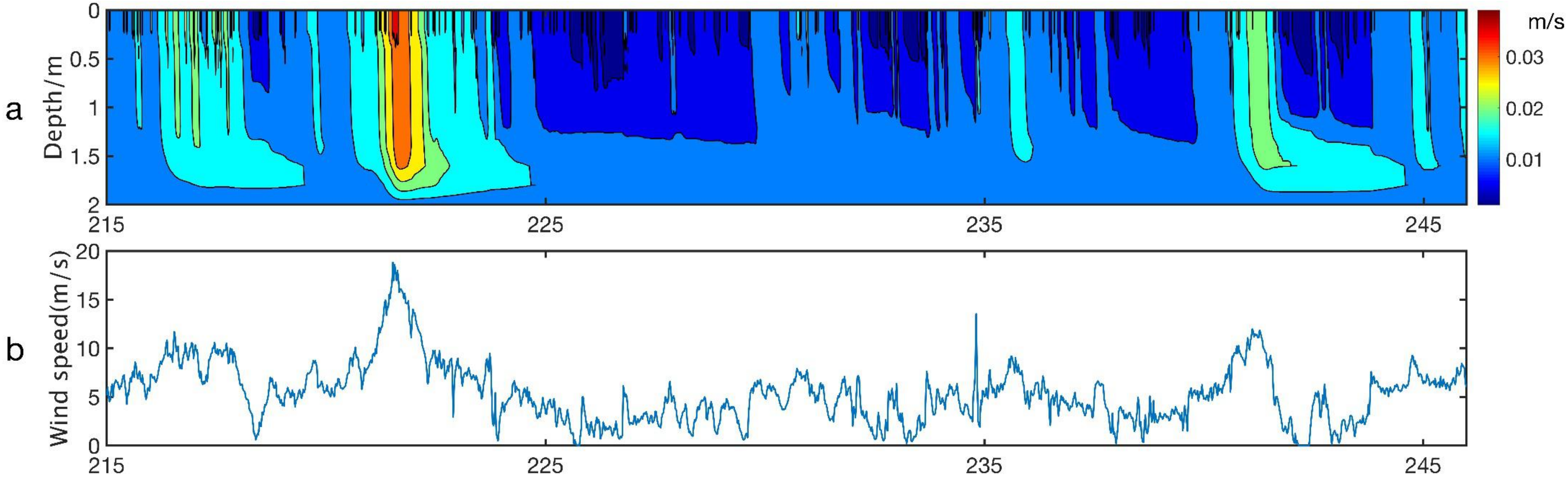
Verified data: Tw, H, LE

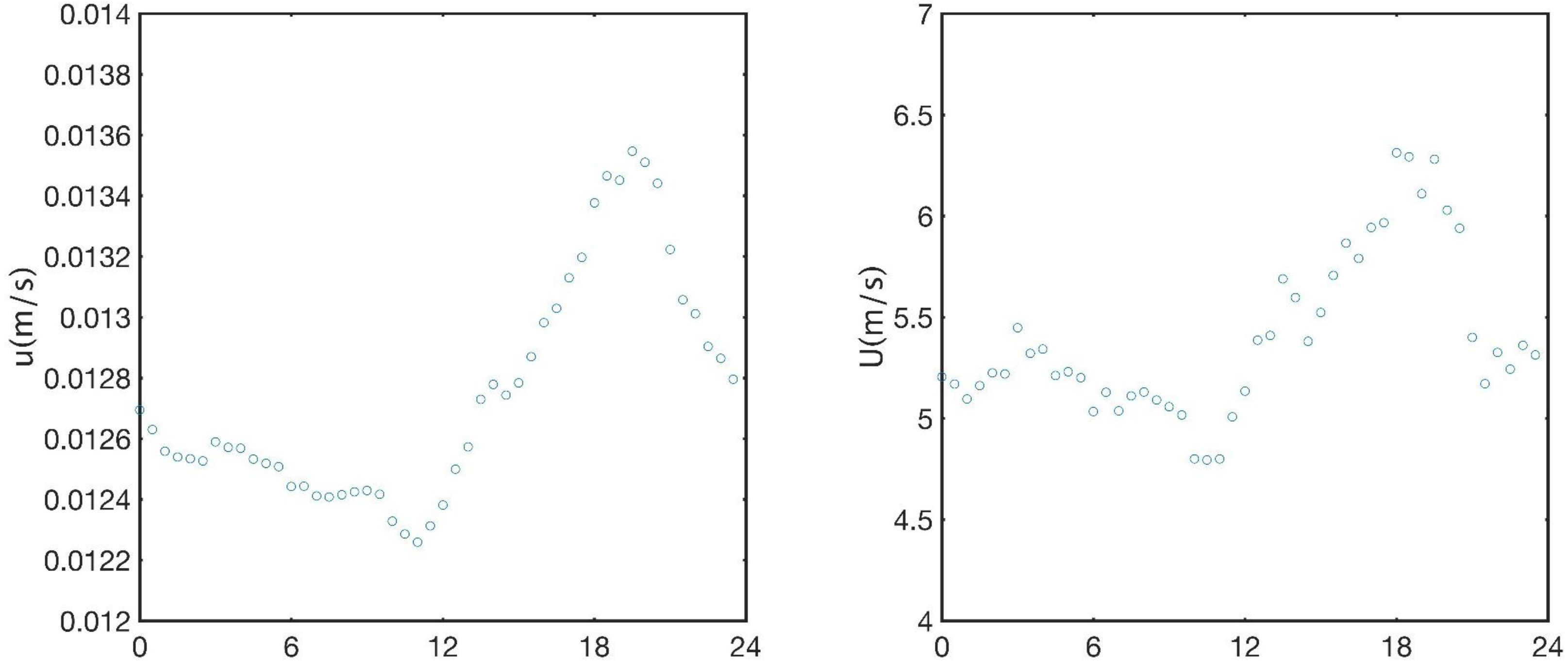
Data: acquired from The Taihu Eddy Flux Network: BFG site from 2012 August

SPINUP Time: 5 Days

Preliminary Result

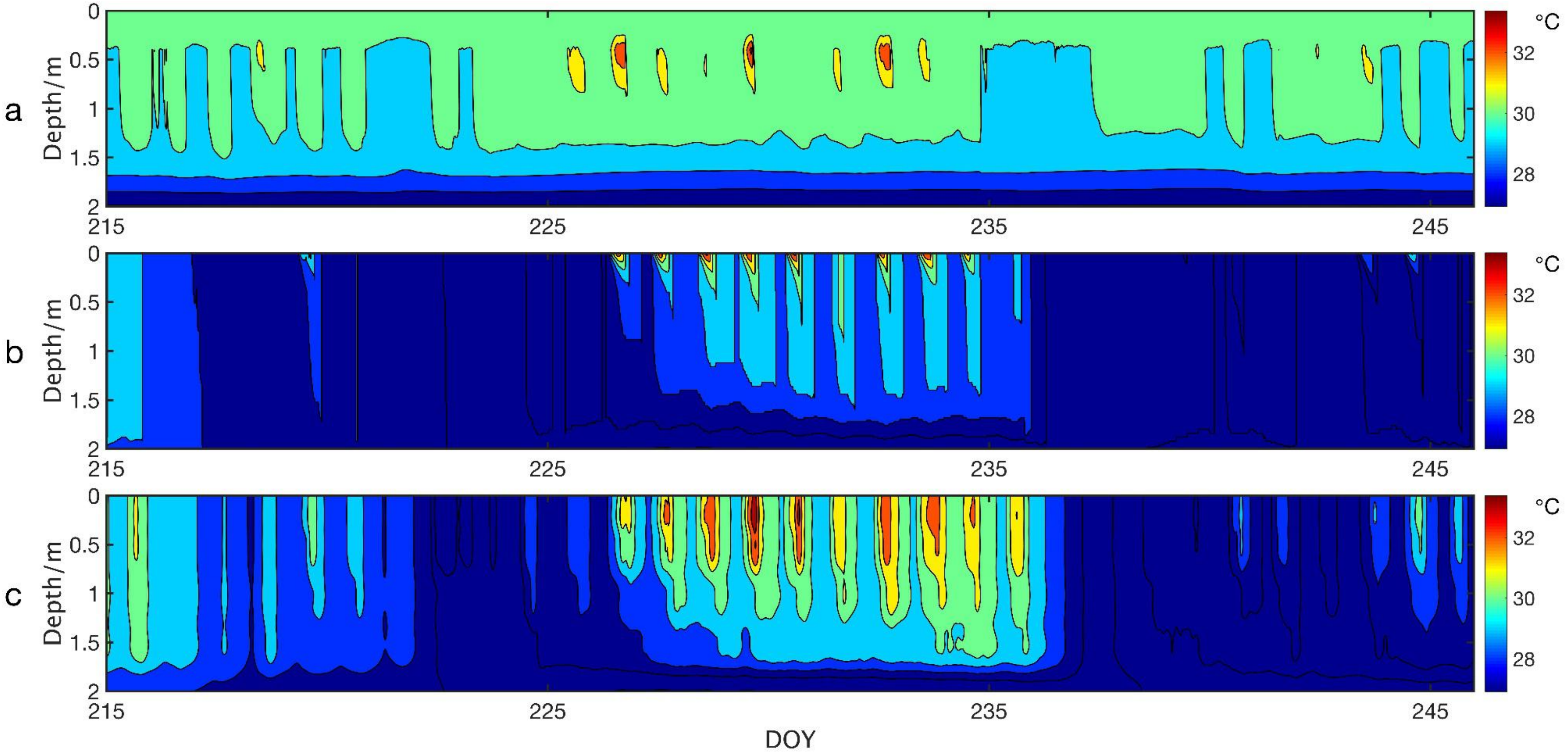
Velocity vector output





u(m/s)
0.014
0.0138
0.0136
0.0134
0.0132
0.013
0.0128
0.0126
0.0124
0.0122
0.012
0
6
12
18
24
U(m/s)
7
6.5
6
5.5
5
4.5
4
0
6
12
18
24

Water temperature output



$a.$ Current revised model
$b.$ $k-\varepsilon$ model
$c.$ $CLM4-LISSS$ model

TKE and dissipation output

```
 LAKE=   2.000000
 tke=  7.821316031587910E-002
 tke=  -68.3955043135140
 tke=  -47150.8361630266
 tke=  -493230115.851480
 tke=   693026853385.677
 tke=  1.133167652527758E+018
 tke=  7.357005621517025E+022
 tke=  5.488495089924519E+030
 tke= -5.714372283456401E+037
 tke=  2.163727422720262E+045
 tke= -6.602529894509531E+053
 tke=  1.426490320091164E+062
 tke=  6.491858265426238E+070
 tke= -2.581357738851519E+080
 tke= -2.334184825128171E+090
 tke=  5.548353362204589E+101
 tke=  1.802485306297514E+113
 tke= -2.282272966498388E+125
 tke= -7.492101201643931E+138
 tke=  4.246573006565100E+153
 tke=  1.781620573628865E+169
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
 tke=                      NaN
```

```
 LAKE=   2.000000
 dissipation rate=  0.150112343445487
 dissipation rate=  -63.7250905827027
 dissipation rate=   426318.708983271
 dissipation rate=  -5264597347.30376
 dissipation rate= -1.243470195608932E+015
 dissipation rate=  6.289972855390491E+020
 dissipation rate= -7.512013454928589E+027
 dissipation rate=  1.294987156324653E+034
 dissipation rate= -1.723357904930746E+042
 dissipation rate=  6.362837331216615E+050
 dissipation rate= -1.086110451619707E+059
 dissipation rate= -1.363811392059899E+068
 dissipation rate=  1.706784330449785E+077
 dissipation rate=  6.490686324320469E+087
 dissipation rate= -6.841836465790523E+098
 dissipation rate= -4.076496158113761E+110
 dissipation rate=  5.259827645958008E+122
 dissipation rate=  1.522086943132161E+136
 dissipation rate= -8.252564333759664E+150
 dissipation rate= -3.180821573981891E+166
 dissipation rate=  1.069428872668477E+183
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
 dissipation rate=                      NaN
```

To be continue…

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x} + \frac{\partial}{\partial z}\left((\nu+\nu_t)\frac{\partial u}{\partial z}\right),$$

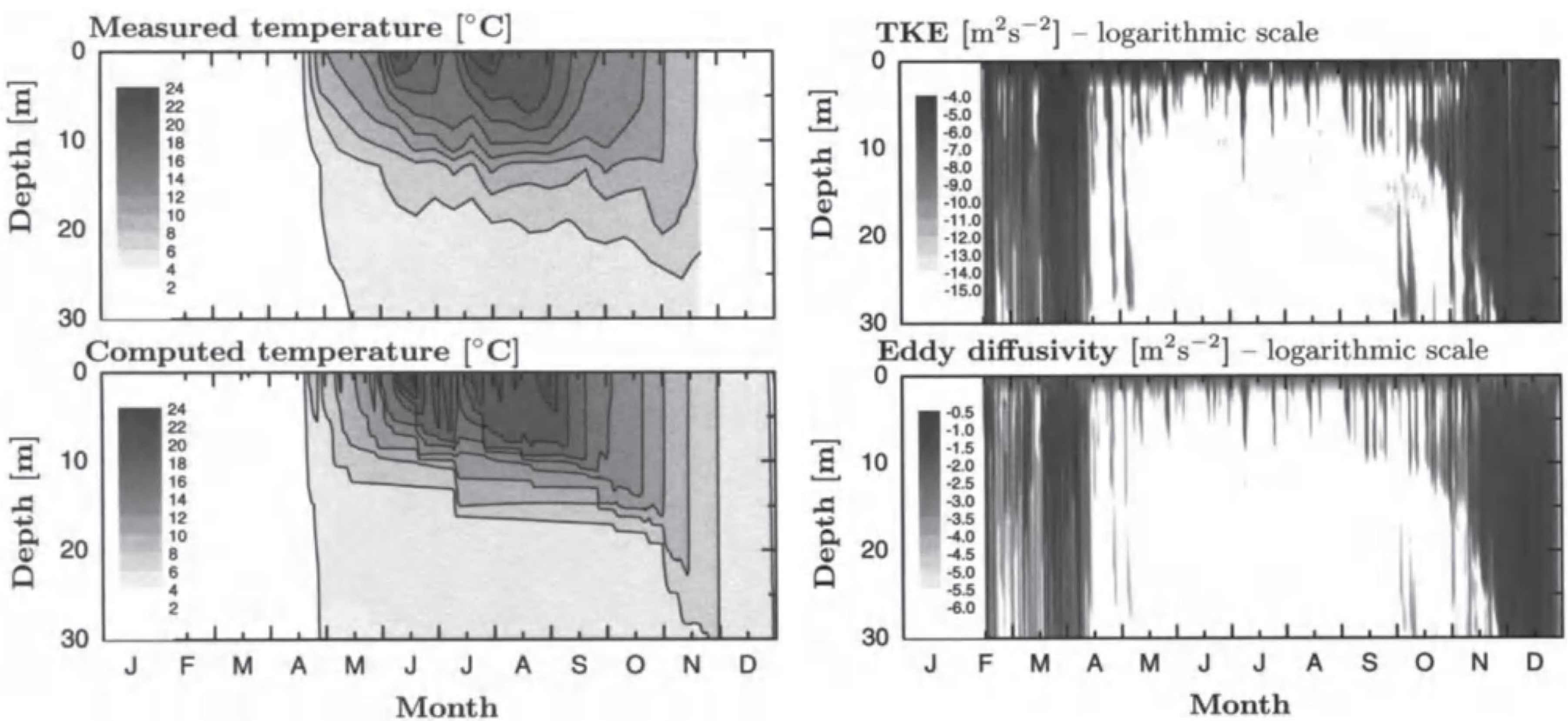
$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial y} + \frac{\partial}{\partial z}\left((\nu+\nu_t)\frac{\partial v}{\partial z}\right),$$

$$0 = \frac{\partial\langle p\rangle}{\partial z} + \langle\rho\rangle g\,,$$

$$\frac{\partial\Theta}{\partial t} = \frac{\partial}{\partial z}\left(\left(\chi^{(\Theta)} + \frac{\nu_t}{\sigma_\Theta}\right)\frac{\partial\Theta}{\partial z}\right) + \frac{1}{\rho_0 c_p}\frac{\partial I}{\partial z}$$

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial z}\left(\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial z}\right) + \frac{c_\mu k^2}{\varepsilon}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right) - \frac{\langle\rho\rangle\langle\alpha_\Theta\rangle}{\rho_0}\frac{\nu_t}{\sigma_\Theta}g\frac{\partial\Theta}{\partial z} - \varepsilon\,,$$

$$\frac{\partial\varepsilon}{\partial t} = \frac{\partial}{\partial z}\left(\left(\nu+\frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial z}\right) + c_1 k\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right) - c_3\frac{\langle\rho\rangle\langle\alpha_\Theta\rangle}{\rho_0}\frac{\nu_t}{\sigma_\Theta}g\frac{\partial\Theta}{\partial z}\cdot\frac{\varepsilon}{k} - \frac{c_2\varepsilon^2}{k}\,.$$



(Hutter 2003)

Summary

The simulation of current revised model need to be checked repeatedly

Clear the boundary conditions of differential equation

Furtherly optimize the algorithm of tridiagonal matrix

Thank you