Yale University
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Coordinate Rotation and Flux Bias Error

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Presentation Outline

1. Why coordinate rotation
2. Theoretical assessment of flux bias errors
3. Examples of incorrect coordinate
4. Three coordinate systems
5. Analysis of a sample dataset

Reasons for performing coordinate rotation

Kaimal and Haugen (1969): " … in perfectly level terrain the anemometers should be leveled to within 0.1 degree"

Kaimal and Finnigan (1994): " … problems occur when vector quantities like velocities or fluxes are measured in a reference framework that does not coincide with that of the equations used to analyze them"

Equations used for analyzing eddy covariance data over level terrain

Conservation of momentum

$$\partial\overline{u}_i/\partial t + \overline{u}_j\partial\overline{u}_i/\partial x_j =$$
$$-\partial(\overline{u_i'u_j'})/\partial x_j - \rho^{-1}\partial\overline{p}/\partial x_i - g\delta_{i3} - 2\Omega\epsilon_{ijk}\eta_j\overline{u}_k$$

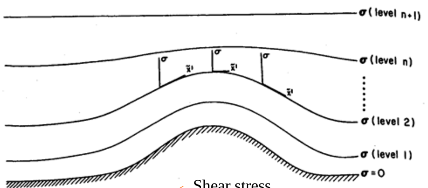
Conservation of mass

$$\partial\overline{c}/\partial t + \overline{u}_j\partial\overline{c}/\partial x_j + \partial(\overline{u_j'c'})/\partial x_j = s$$

$$\text{NEE} \equiv \int_0^h s\,\mathrm{d}z + (\overline{w'c'})_{z=0}$$

$$= \int_0^{z_r} \frac{\partial\overline{c}}{\partial t}\mathrm{d}z + (\overline{w'c'})_{z_r}$$

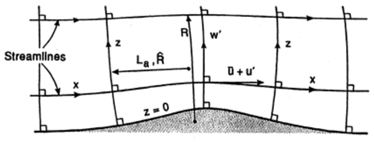
Terrain-following coordinate
(Pielke 1984)



Shear stress

$$\frac{\partial\bar{\tilde{u}}^1}{\partial t} = -\bar{\tilde{u}}^j\frac{\partial\bar{\tilde{u}}^1}{\partial\tilde{x}^j} - \boxed{\overline{\tilde{u}^{j''}\frac{\partial\tilde{u}^{1''}}{\partial\tilde{x}^j}}} - \bar{\theta}\frac{\partial\bar{\pi}}{\partial\tilde{x}^1} + g\frac{\sigma - s}{s}\frac{\partial z_G}{\partial x} - \hat{f}\bar{\tilde{u}}^3 + f\bar{\tilde{u}}^2,$$
$$\frac{\partial\bar{\tilde{u}}^2}{\partial t} = -\bar{\tilde{u}}^j\frac{\partial\bar{\tilde{u}}^2}{\partial\tilde{x}^j} - \boxed{\overline{\tilde{u}^{j''}\frac{\partial\tilde{u}^{2''}}{\partial\tilde{x}^j}}} - \bar{\theta}\frac{\partial\bar{\pi}}{\partial\tilde{x}^2} + g\frac{\sigma - s}{s}\frac{\partial z_G}{\partial y} - f\bar{\tilde{u}}^1.$$

Streamline coordinate
(Finnigan 1983, 1999)



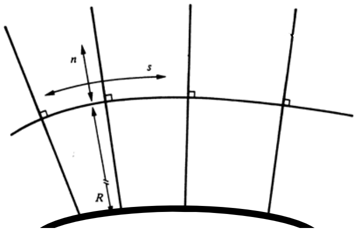
Conservation of momentum

$$\overline{u}\frac{d\overline{u}}{dx} = -\frac{1}{\rho}\frac{d\overline{p}}{dx} - \frac{d\overline{u'^2}}{dx} + \left(\frac{\overline{u'^2} - \overline{w'^2}}{L_a}\right) - \frac{d}{dz}(\overline{u'w'})$$
$$+ 2\left(\frac{\overline{u'w'}}{R}\right) + \frac{\overline{u'w'}}{L_b} + \left(\frac{\overline{w'^2} - \overline{v'^2}}{L_c}\right) + g_x\left(\frac{\overline{\theta} - \overline{\theta}_0}{\overline{\theta}_0}\right)$$

Conservation of mass

$$\frac{\partial\bar{c}}{\partial t} + \bar{u}\partial_x\bar{c} = -\partial_x\overline{u'c'} - \frac{\overline{u'c'}}{L_a} - \partial_z\overline{w'c'} + \frac{\overline{w'c'}}{R} + \bar{s}(x,z)$$

Surface-following coordinate
(Howarth 1951)



Surface-following coordinate
(Howarth 1951)

s-component momentum equation

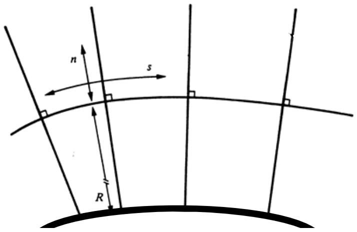
$$\overline{u}\frac{\partial\overline{u}}{\partial s} + (1 + \frac{n}{R})\,\overline{w}\,\frac{\partial\overline{u}}{\partial n} + \frac{\overline{u}\,\overline{w}}{R} =$$

$$-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial s} - \frac{\partial\overline{u'^2}}{\partial s} - (1 + \frac{n}{R})\frac{\partial\overline{u'w'}}{\partial n} - 2\frac{\overline{u'w'}}{R}$$

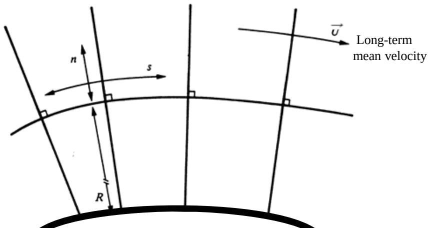
Simplified s-component momentum equation

$$\overline{u}\frac{\partial\overline{u}}{\partial s} + \overline{w}\frac{\partial\overline{u}}{\partial n} = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial s} - \frac{\partial\overline{u'w'}}{\partial n}$$

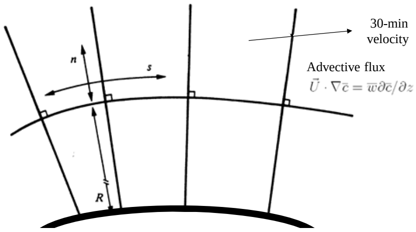
Surface-following coordinate



Surface-following coordinate



Surface-following coordinate



Advective flux

$\vec{U}\cdot\nabla\overline{c} = \overline{w}\partial\overline{c}/\partial z$

Momentum flux error due to sensor tilt

Flux in tilted coordinate

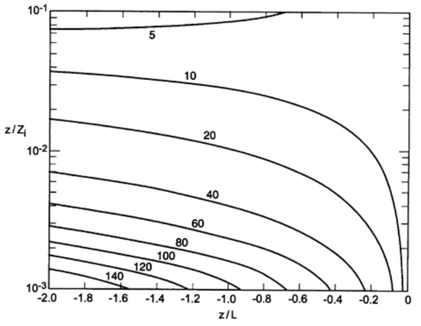
True flux

$$\boxed{\overline{u_1'w_1'}} = \boxed{\overline{u'w'}}\cos(2\alpha) + \frac{1}{2}(\overline{w'^2} - \overline{u'^2})\sin(2\alpha)$$

$$\sigma_u/u_* = (12 - 0.5Z_i/L)^{1/3}$$

$$\sigma_w/u_* = 1.25(1 - 3z/L)^{1/3}$$

Percent of momentum flux error as a function of *z/L* and *z/Z*$_i$ for a tilt angle of 1 degree (Wilczak et al. 2001)



Scalar flux error due to sensor tilt

Flux in tilted coordinate

True flux

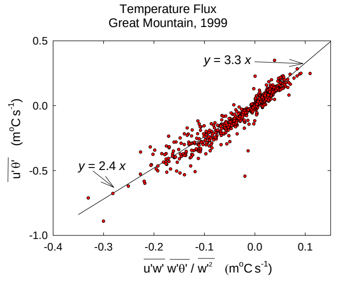
$$\boxed{\overline{w_1'c'}} = \boxed{\overline{w'c'}}\cos(\alpha) + \overline{u'c'}\sin(\alpha)$$

$$= \overline{w'c'}\cos(\alpha) + a\frac{\overline{u'w'}}{\overline{w'^2}}\overline{w'c'}\sin(\alpha)$$

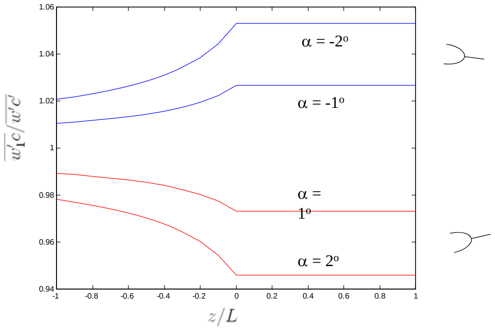
$$\sigma_w/u_* = 1.25(1-3z/L)^{1/3}$$

Generation of Horizontal Eddy Flux

$$\overline{u'c'} = -\tau_{ux}\overline{u'^2}\frac{\partial\bar{c}}{\partial x} + \alpha\frac{\overline{u'w'}\;\overline{w'c'}}{\overline{w'^2}}$$



Scalar flux error due to sensor tilt

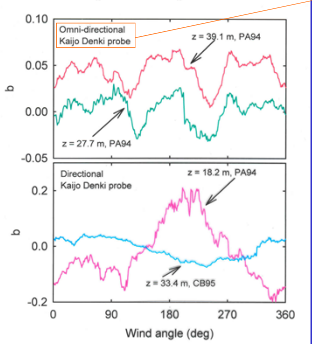


Incorrect Coordinate Systems

- Instrument tilt
- Offset in instrument vertical velocity
- Non-zero mean vertical velocity due to mesoscale motion
- Aerodynamic interference by tower and anemometer frame

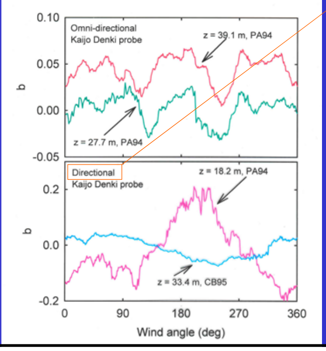
Aerodynamic interference caused by anemometer's supporting frame

$$\overline{w}_1 = a + b\overline{u}_1 + \overline{w}$$



Aerodynamic interference caused by anemometer’s supporting frame

$$\overline{w}_1 = a + b\overline{u}_1 + \overline{w}$$



Aerodynamic interference caused by anemometer’s supporting frame

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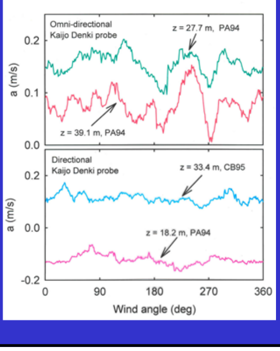

Natural wind coordinate system
(Tanner and Thurtell 1969)

Natural wind coordinate system:
A right-handed coordinate system in which the x-axis is parallel to the mean flow, with the mean vertical velocity, lateral velocity and the v-w covariance set to zero

Problems:
1) Sensitive to instrument offset particularly at low wind speed
2) Covariance between w-v may be nonzero in non-steady state conditions, over nonflat terrain, or over the sea
3) Information on mean vertical velocity is lost

Planar fit coordinate system
(Paw U et al. 2000, Wilczak et al. 2001)

Planar fit coordinate: a right-handed orthogonal coordinate in which the *z*-axis is perpendicular to the plane of the mean streamline and the *x*-axis is parallel to the mean wind direction for each observation

Steps:

1) Determine the period during which there was no change to the anemometer’s position
2) Perform a linear regression using an appropriate subset of the data to define the “tilted plane”, or plane of the mean streamline
3) Use the regression coefficients b_1 and b_2 as in $w_1=b_0 + b_1\, u_1 + b_2\, v_2$ to determine the rotation angles
4) Rotate the velocities and covariance terms to the new coordinate

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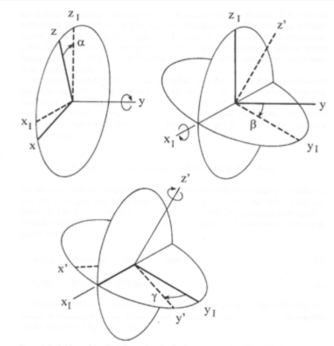
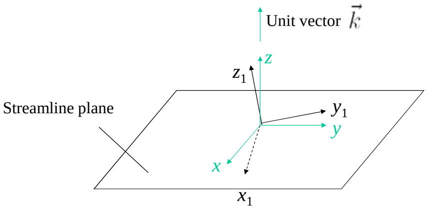
Three-step coordinate rotation scheme
(Wilczak et al. 2001)



Planar fit coordinate system



Vertical velocity in planar fit coordinate

$$\begin{aligned} w &= \vec{k} \cdot \vec{U} \\ &= k_{x_1} u_1 + k_{y_1} v_1 + k_{z_1}(w_1 - b_0) \end{aligned}$$

Or $$w_1 = b_0 \overset{b_1}{\boxed{- \frac{k_{x_1}}{k_{z_1}}}} u_1 \overset{b_2}{\boxed{- \frac{k_{y_1}}{k_{z_1}}}} v_1 + \overset{\text{“Random noise”}}{\boxed{\frac{1}{k_{z_1}} w}}$$

Vertical flux $$\overline{w'c'} = k_{x_1} \overline{u_1'c'} + k_{y_1} \overline{v_1'c'} + k_{z_1} \overline{w_1'c'}$$

Unit vector parallel to the y-axis $$\vec{j} = \frac{\vec{k} \times \vec{U}}{|\vec{k} \times \vec{U}|}$$

Unit vector parallel to the x-axis $$\vec{i} = \vec{j} \times \vec{k}$$

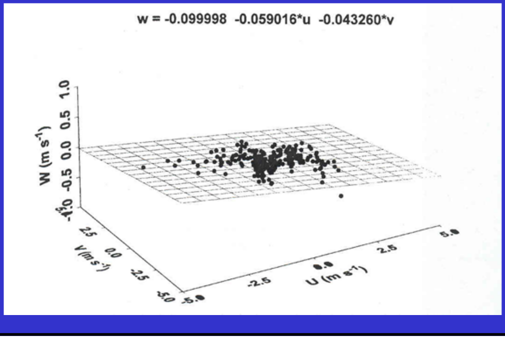
```
% function [k,b0]=unit_vector_k(U1)
% determines unit vector k (parallel to new z-axis)
% input
%    U1(:,1): mean u1 in instrument coordinate
%      (:,2): mean v1 in instrument coordinate
%      (:,3): mean w1 in instrument coordinate
% output
%   k: unit vector parallel to new coordinate z axis
%   b0: instrument offset in w1

function [k,b0]=unit_vector_k(U1)
% wilzack's routine
u=(U1(:,1))'; v=(U1(:,2))'; w=(U1(:,3))';
flen=length(u);
su=sum(u); sv=sum(v); sw=sum(w); suv=sum(u*v'); suw=sum(u*w');
svw=sum(v*w'); su2=sum(u*u'); sv2=sum(v*v');
H=[flen su sv; su su2 suv; sv suv sv2]; g=[sw suw svw]';
x=H\g; b0=x(1); b1=x(2); b2=x(3);
% determine unit vector k
k(3)=1/(1+b1^2+b2^2);
k(1)=-b1*k(3);
k(2)=-b2*k(3);
return;
```

```
% function [i,j]=unit_vector_ij(U1,k)
%
% determines unit vectors i, j (parallel to new coordinate x and y axes
%
% input
%    U1(1): (30-min) mean u1 in instrument coordinate
%      (2):  v1 in instrument coordinate
%      (3):  w1 in instrument coordinate
%   k: unit vector parallel to the new coordinate x-axis
% output
%   i,j: unit vector parallel to new coordinate x, and y axes

function [i,j]=unit_vector_ij(U1,k)
j=cross(k,U1);
j=j/(sum(j.*j))^0.5;
i=cross(j,k);
return
```

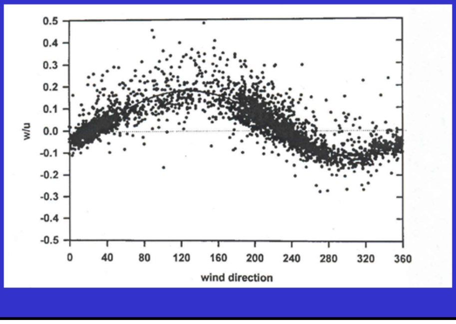
```
% function [uc,vc,wc]=scalar_flux(u1c,v1c,w1c,i,j,k)
%
% determines scalar flux in new coordinate
%
% input
%   u1c,v1c,w1c: scalar flux in instrument coordinate
%   i,j, k: unit vectors parallel to the new coordinate x, y and z-axes
% output
%   uc,vc,wc: scalar flux in new coordinate

function [uc,vc,wc]=scalar_flux(u1c,v1c,w1c,i,j,k)
H=[u1c v1c w1c];
uc=sum(i.*H);
vc=sum(j.*H);
wc=sum(k.*H);
return;
```

```
% function [uu,vv,ww,uw,vw]=velocity_stat(u,i,j,k)
% determines velocity statistics in new coordinate
% input
%   u: 3 by 3 matrix of cross product of the three velocity components
%      (u(1,1) = u1^u1,u(1,2)=u1^v1,etc.) in instrument coordinate
%   i,j, k: unit vectors parallel to the new coordinate x, y and z-axes
% output
%   uu,vv,ww,uw,vw: statistics in new coordinate
function [uu,vv,ww,uw,vw]=velocity_stat(u,i,j,k)
uu=i(1)^2*u(1,1)+i(2)^2*u(2,2)+i(3)^2*u(3,3)+ . . .
   2*(i(1)*i(2)*u(1,2)+i(1)*i(3)*u(1,3)+i(2)*i(3)*u(2,3));
vv=j(1)^2*u(1,1)+j(2)^2*u(2,2)+j(3)^2*u(3,3)+...
   2*(j(1)*j(2)*u(1,2)+j(1)*j(3)*u(1,3)+j(2)*j(3)*u(2,3));
ww=k(1)^2*u(1,1)+k(2)^2*u(2,2)+k(3)^2*u(3,3)+...
   2*(k(1)*k(2)*u(1,2)+k(1)*k(3)*u(1,3)+k(2)*k(3)*u(2,3));
uw=i(1)*k(1)*u(1,1)+i(2)*k(2)*u(2,2)+i(3)*k(3)*u(3,3)+...
   (i(1)*k(2)+i(2)*k(1))*u(1,2)+(i(1)*k(3)+i(3)*k(1))*u(1,3)+...
   (i(2)*k(3)+i(3)*k(2))*u(2,3);
vw=j(1)*k(1)*u(1,1)+j(2)*k(2)*u(2,2)+j(3)*k(3)*u(3,3)+...
   (j(1)*k(2)+j(2)*k(1))*u(1,2)+(j(1)*k(3)+j(3)*k(1))*u(1,3)+...
   (j(2)*k(3)+j(3)*k(2))*u(2,3);
return;
```

Two-dimensional plane regression
Wind River
(Paw U et al. 2000)

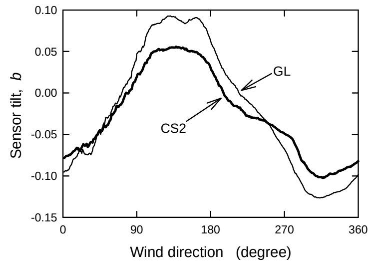


Instrument tilt as a function of wind direction Walker Branch (Baldocchi et al. 2000)

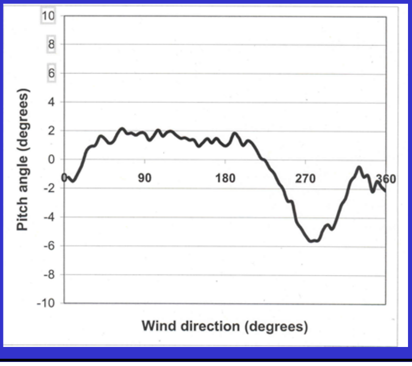


Instrument tilt as a function of wind direction Great Mountain (Lee and Hu 2002)

$$b(\phi) \neq -b(\phi + \pi)$$

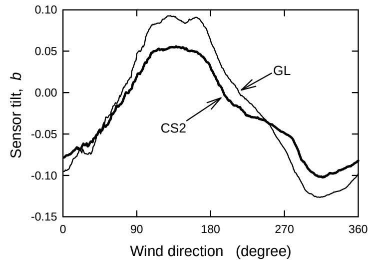


The angle between the mean wind vector and the azimuthal average vertical direction Tumbarumba (Finnigan et al. 2002)



Instrument tilt as a function of wind direction Great Mountain

$$b(\phi) \neq -b(\phi + \pi)$$



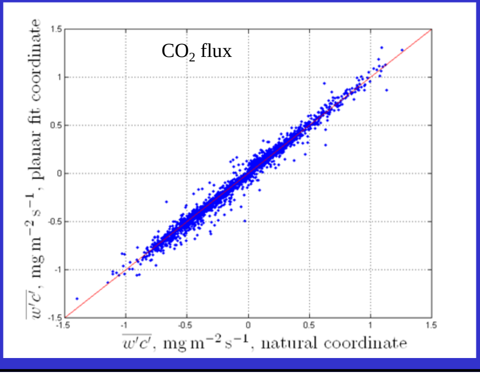
Modified Planar fit (MPF) coordinate system

Modified planar fit coordinate: same as the planar fit coordinate except that the "plane" of the mean streamline is dependent on wind direction

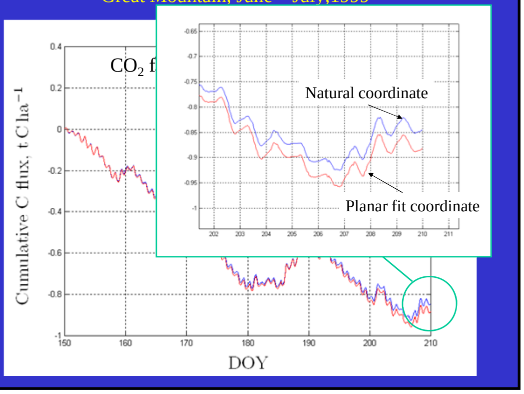
Steps:

1) Determine the period over which there was no change to the anemometer's position
2) Divide wind direction into bins
3) Perform a linear regression using an appropriate subset of the data to define the "tilted plane" for each wind direction bin
4) Use the regression coefficients a_1 and a_2 as in $w_1=a_0 + a_1\, u_1 + a_2\, v_2$ to determine the rotation angles
5) Rotate the velocities and covariance terms to the new coordinate

Comparison of Natural and Planar Fit Coordinates Great Mountain, June – July,1999



Comparison of Natural and Planar Fit Coordinates
Great Mountain, June – July,1999



Comparison of Natural and Planar Fit Coordinates
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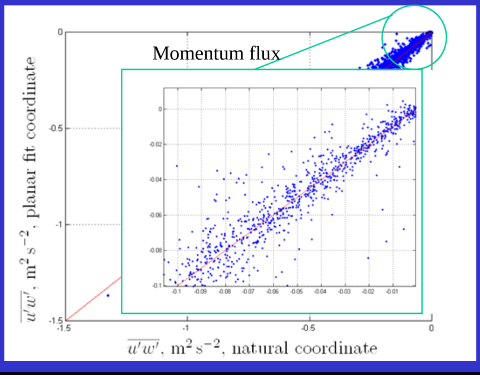
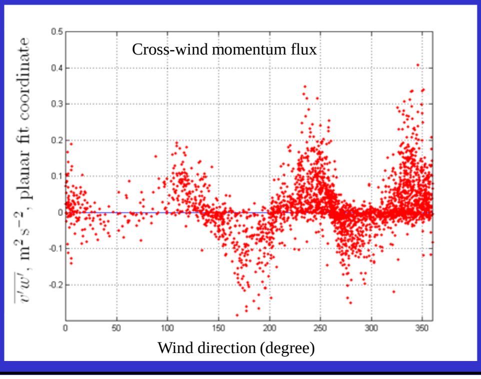
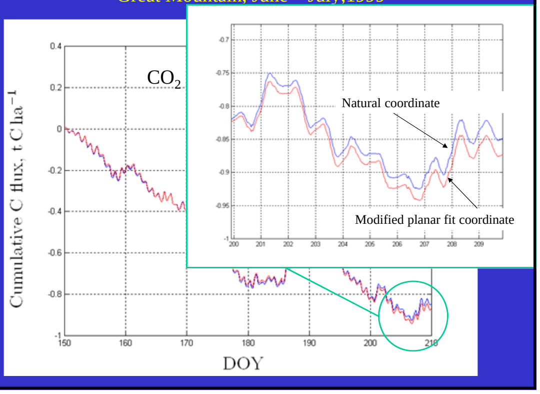


Comparison of Natural and Planar Fit Coordinates
Great Mountain, June – July,1999



Comparison of Natural and Modified Planar Fit Coordinates
Great Mountain, June – July,1999



Comparison of Natural and Modified Planar Fit Coordinates
Great Mountain, June – July,1999